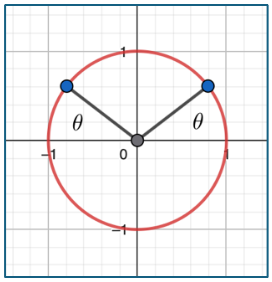
Finding All Angles with a Given Sine or Cosine

Chuck Allison, April 2025

If $\sin\theta = y$, we know that $\sin(\theta + 2n\pi) = y$ for all integers n. But it is also true that $\sin\theta = \sin(\pi - \theta)$:



SINES ARE THE SAME

Therefore, the following angles have the same sine, y:

1. $\theta + 2n\pi$
2. $(\pi - \theta) + 2n\pi = (2n + 1)\pi - \theta$

where n is *any* integer. For any of these angles, α, we have $\sin^{-1} y = \alpha$, if we "extend" $arcsin$ to a **relation**.

If $\cos\theta = y$, we know that $\cos\theta = \cos(-\theta)$, therefore, the following angles have the same cosine, y:

3. $\theta + 2n\pi$
4. $-\theta + 2n\pi$

For any such angles, α, $\cos^{-1} y = \alpha$, for an "extended" $arccos$ relation.

Reducing Angles to Fundamental Intervals

The **fundamental interval** for $\sin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the interval nearest the origin that contains all the values of the sine: $[-1,1]$. For values of x outside that interval, we can reduce x to a value $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin x = \pm \sin t$.

The idea is to subtract a multiple of π (not 2π; otherwise, we might skip over the fundamental interval altogether!) as follows:

$$t = x - n\pi$$

where the integer n **minimizes** $|t|$. That will guarantee that t will be as close to zero as possible, thus placing it in the desired interval.

The value of n that minimizes $|t| = |x - n\pi|$ will also minimize $\left|\frac{t}{\pi}\right| = \left|\frac{x-n\pi}{\pi}\right| = \left|\frac{x}{\pi} - n\right|$. The integer, n, that minimizes the latter expression is the *closest* integer to $\frac{x}{\pi}$, so we *round* that quantity to the *nearest* integer to obtain n, and we have $n = \left[\frac{x}{\pi}\right] = \frac{x}{\pi} + \delta,\ |\delta| \leq \frac{1}{2}$, therefore

$$|t| = |x - n\pi| = \left|x - \left(\frac{x}{\pi} + \delta\right)\pi\right| = |\delta|\pi \leq \frac{1}{2}\pi$$

which guarantees $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

We now have $t = x - n\pi$, but $\sin t$ will equal $\sin x$ only if n is **even**. If n is **odd**, then $\sin x = -\sin t$, giving the formula

$$\boldsymbol{\sin x = (-1)^{|n|} \sin t.}$$

For **cosine** the desired interval is $[0, \pi\,]$. Instead of rounding to the *nearest* integer, we will use the **floor** function to round *down*:

$$n = \left\lfloor\frac{x}{\pi}\right\rfloor = \frac{x}{\pi} - \delta, \qquad 0 \leq \delta < 1$$

We now see that $|t| = |x - n\pi| = \left|x - \left(\frac{x}{\pi} + \delta\right)\pi\right| = \delta\pi < \pi$, giving $t \in [0, \pi)$.

If n is **even**, then $\cos x = \cos t$. If n is **odd**, then $\cos x = -\cos t = \cos(\pi - t)$, so we have

$$\boldsymbol{\cos x} = \begin{cases} \boldsymbol{\cos t}, & n = 2k \\ \boldsymbol{\cos(\pi - t)}, & n = 2k+1 \end{cases}$$

For the case where $\delta = 0 \Rightarrow x = n\pi$ we have $\cos x = (-1)^{|n|}$, since $|\cos n\pi| = 1$.

Put another way:

$$\boldsymbol{\sin^{-1}(\sin x)} = \begin{cases} \boldsymbol{t}, & n = 2k \\ \boldsymbol{-t}, & n = 2k+1 \end{cases}$$

$$\boldsymbol{\cos^{-1}(\cos x)} = \begin{cases} \boldsymbol{(-1)^{|n|}}, & x = n\pi \\ \boldsymbol{t}, & n = 2k \\ \boldsymbol{\pi - t}, & n = 2k+1 \end{cases}$$

where t is within the fundamental interval for the associated inverse trigonometric function.